Complex Variables

Test 3

Instructions: The usual.

1. Use residues to evaluate the following integrals.

   (a)
   $$\int_{-\infty}^{\infty} \frac{x}{(x^2+4x+13)^2} dx$$

   (b)
   $$\int_{0}^{\infty} \frac{x^2}{x^4+1} dx$$

2. Use residues to evaluate the following integrals.

   (a)
   $$\int_{0}^{\infty} \frac{x\sin(x)}{x^4+1} dx$$

   (b)
   $$\int_{0}^{\infty} \frac{\cos(x)-1}{x^2+1} dx$$

3. Show that in polar coordinates, a function u is harmonic exactly when
   $$r\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0$$

4. Suppose that u is non-negative and harmonic on all of $\mathbb{C}$. Show that u is constant.

5. Suppose f is holomorphic in a region containing the upper half plane $\{z = x+iy \in \mathbb{C} : y \geq 0\}$ and tends to zero as $z \to \infty$. Then for $z = x+iy \in \mathbb{C}$ with $(y > 0)$, we have
   $$f(z) = \frac{1}{2\pi i}\int_{\infty}^{\infty} \frac{f(t)}{t-z} dt$$
   $$0 = \frac{1}{2\pi i}\int_{\infty}^{\infty} \frac{f(t)}{t-\bar{z}} dt$$

   Use this to show Poisson's formula for the upper half plane:
   If u is harmonic in a region containing $\{z = x+iy \in \mathbb{C} : y \geq 0\}$ and tends to zero as $z \to \infty$, then for $z = x+iy \in \mathbb{C}$ with $(y > 0)$, we have
   $$u(x,y) = \frac{y}{\pi}\int_{-\infty} \infty \frac{u(t,0)}{(x-t)^2+y^2} dt$$

   (You can use without proof that u will be the real part of a holomorphic function which tends to zero as $z \to \infty$.)